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Quantum Poincaré algebra with standard real structure

Jerzy Lukierski [1]
Institute for Theoretical Physics, University of Wrocław, pl. Maxa Borna 9, 50-204 Wrocław, Poland

Anatol Nowicki [1]
Institute of Physics, Pedagogical University, Pl. Słowiański 6, 65-029 Zielona Góra, Poland

Henri Ruegg [2]
Departement de Physique Théorique, Université de Genève, 24, quai Ernest-Ansermet, CH-1211 Geneva 4, Switzerland

A new real quantum Poincaré algebra which is a standard *-Hopf algebra is obtained by the construction of $U_q(\mathrm{O}(3,2))$ (q real). The deformation parameter κ is mass-like, and the classical Poincaré algebra is obtained in the limit $\kappa \to \infty$. For our κ-Poincaré algebra both Casimirs are given.

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Contents

1. Introduction

Recently the present authors considered quantum deformations of the $D = 4$ Poincaré algebra refs. [1–4] obtained by the contraction of a real form of the

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quantum anti-de Sitter algebra $U_q(O(3,2))$. The method was based on finding the deformation of the Cartan–Weyl basis, with generators self-conjugate with respect to involutive homomorphisms (involutions) describing the real structure. In ref. [2] we considered all inner involutions of the Cartan–Weyl basis for $U_q(\mathrm{Sp}(4;\mathbb{C}))$. We found in ref. [2] that only two of them (out of sixteen) provide examples of real forms of $U_q(\mathrm{Sp}(4;\mathbb{C}))$ suitable for our contraction procedure to the quantum Poincaré algebra [#1]. Unfortunately these two real forms were described by nonstandard involutions (one $\oplus$-involution, used in ref. [1], and one $*$-involution listed in ref. [2]) [#2].

The standard $+$-involution should be an antiautomorphism in the algebra sector and an automorphism in the coalgebra sector [5–7],

$$(a_1 \cdot a_2)^+ = a_2^+ a_1^+ , \qquad (\Delta(a))^+ = \Delta(a^+) . \tag{1.1}$$

In order to introduce the standard involution defining the real form of $U_q(\mathrm{Sp}(4))$ suitable for our contraction procedure we are forced to consider the involutions which take us out of the Cartan–Weyl basis. [#3] It appears that one should consider the antipode-extended Cartan–Weyl basis $\{h_i, e_{\pm i}, e_{\pm A}, S(e_{\pm A})\}$, where $\{h_i, e_{\pm i}\}$ describe Cartan–Chevalley generators, and $e_{\pm A}$ describe the generators corresponding to nonsimple roots (for Sp(4): $i = 1,2$; $A = 3,4$). In such a case the relations expressing physical real generators (in our case q-deformed O(3,2) generators) in terms of the antipode-extended Cartan–Weyl basis contains additional freedom (in our case we express 10 physical generators in terms of 14 generators). In this paper we shall make a choice which after the contraction [#4] (we recall that q is real)

$$R \to \infty , \qquad R \log q \to \kappa^{-1} \quad (0 < \kappa < \infty) \tag{1.2}$$

provides a new quantum (κ-deformed) Poincaré algebra, with standard real structure.

We would like to point out here that the contraction limit of $U_q(O(3,2))$ given in ref. [1] after a suitable nonlinear transformation of the κ-deformed boost generators (see ref. [10]) provides the simplified form of the real quantum Poincaré algebra. Interestingly enough, the contraction limits in ref. [1] (for $|q| = 1$) and given in this paper (q real), which look quite different, turn out after suitable nonlinear transformations to be related simply by the replacement $\kappa \to i\kappa$. It should be stressed, however, that in this paper we avoid at least three

#1 We required in refs. [1–3] that the nonrelativistic O(3) rotations as a quantum subalgebra of the quantum Poincaré algebra remains undeformed.

#2 We use the notation described in ref. [2].

#3 In fact, these inner involutions of $U_q(\mathrm{Sp}(4;\mathbb{C}))$ were considered in ref. [2] (see ref. [2], formula (3.22)) but because they were not inner in the Cartan–Weyl basis they were not elaborated.

#4 The contraction (1.2) was firstly introduced by the Firenze group [8,9].

difficulties related with nonstandard $\oplus$-involutions used in our earlier work on real Poincaré algebras (see refs. [1,5], also ref. [10]), namely:

(i) The reality condition for the coproduct ($\Delta' = \tau \circ \Delta$)

$$(\Delta(a))^{\oplus} = \Delta'(a^{\oplus}) \tag{1.3}$$

implies that the real spectrum of the algebra becomes complex on tensor products (e.g., the total three-momentum of the two independent subsystems becomes complex).

(ii) If one wishes to define $\oplus$-involution as an adjoint operation in the representation space $|x\rangle$ endowed with scalar product $\langle x'|x\rangle$ it should be defined on the tensor product $|x_1, x_2\rangle = |x_1\rangle \otimes |x_2\rangle$ as follows:

$$\langle x'_1, x'_2 | x_1, x_2\rangle = \langle x'_1|x_2\rangle\langle x'_2|x_1\rangle . \tag{1.4}$$

The scalar product (1.4) is not positive definite.

(iii) It is not known how to describe the dual objects to $\oplus$-real quantum Lie algebras, which would describe standard real quantum groups.

It should be added that at present two versions of q-deformed Poincaré algebra were proposed:

(i) The one discussed in this paper, with commuting four-momenta and Lorentz generators not forming a quantum subalgebra (see refs. [1–4]).

(ii) The one with the four-momenta forming a quadratic algebra and the Lorentz generators forming a quantum subalgebra. Such a structure was obtained from q-deformation of the $D = 4$ conformal algebra [11,12] or from the realization of q-differential calculus on q-deformed Minkowski space [13].

The first approach has its advantage that the q-Poincaré algebra is a genuine Hopf algebra, with coproducts embedded in the tensor product of q-Poincaré enveloping algebras. The second approach leads to the desirable property that the q-Lorentz algebra is a Hopf subalgebra, but it is rather a quantum Weyl than a quantum Poincaré algebra (eleventh scaling generator is needed for defining the coproducts for ten q-Poincaré generators).

2. Standard real quantum algebras $U_q(\mathrm{O}(3,2))$

Firstly we recall the basic formulae defining the Cartan–Weyl basis of $U_q(\mathrm{Sp}(4;\mathbb{C}))$ [1,2]. Introducing the symmetrized Cartan matrix for the Lie algebra $C_2 \equiv \mathrm{Sp}(4)$,

$$\alpha_{ij} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} , \tag{2.1}$$

the quantum Lie algebra $U_q(\mathrm{Sp}(4;\mathbb{C}))$ is described by the following Cartan–Chevalley generators, corresponding to the simple roots of C_2 $(i = 1, 2)$:

$$[e_i, e_{-j}] = \delta_{ij}\,[h_j]_q\,,\qquad [h_i, e_{\pm j}] = \pm\alpha_{ij}e_j\,,\qquad [h_i, h_j] = 0\,, \tag{2.2}$$

restricted by the following q-Serre relations:

$$[e_{\pm 1}, [e_{\pm 1}, [e_{\pm 1}, e_{\pm 2}]_{\tilde{q}^{\mp 1}}]_{\tilde{q}^{\mp 1}}]_{\tilde{q}^{\mp 1}} = 0\,,\qquad [e_{\pm 2}, [e_{\pm 2}, e_{\pm 1}]_{\tilde{q}^{\mp 1}}]_{\tilde{q}^{\mp 1}} = 0\,, \tag{2.3}$$

where $[e_\alpha, e_\beta]_{\tilde{q}} \equiv e_\alpha e_\beta - q^{-\langle\alpha,\beta\rangle} e_\beta e_\alpha$, The coproduct and antipodes are given by the formulae

$$\Delta(h_i) = h_i \otimes I + I \otimes h_i\,,\qquad \Delta(e_{\pm i}) = e_{\pm i} \otimes k_i + k_i^{-1} \otimes e_{\pm i}\,, \tag{2.4}$$

where $k_i = q^{h_i/2}$ and

$$S(h_i) = -h_i\,, \qquad S(e_{\pm i}) = -q^{\pm\frac{1}{2}d_i} e_{\pm i}\,, \tag{2.5}$$

where $d_i = (1, 2)$.

The Cartan–Weyl basis is defined as follows:

$$e_3 = e_1 e_2 - q e_2 e_1\,, \qquad e_{-3} = e_{-2}e_{-1} - q^{-1} e_{-1} e_{-2}\,,\qquad e_4 = [e_1, e_3]\,, \qquad e_{-4} = [e_{-3}, e_{-1}]\,. \tag{2.6a}$$

Introducing

$$\tilde{e}_3 = e_2 e_1 - q e_1 e_2\,, \qquad \tilde{e}_{-3} = e_{-1} e_{-2} - q^{-1} e_{-2} e_{-1}\,,\qquad \tilde{e}_4 = [\tilde{e}_3, e_1]\,, \qquad \tilde{e}_{-4} = [e_{-1}, \tilde{e}_{-3}]\,, \tag{2.6b}$$

the formulae for the antipode take the form

$$S(e_{\pm 3}) = q^{\pm 3/2}\tilde{e}_{\pm 3}\,, \qquad S(e_{\pm 4}) = -q^{\pm 2}\tilde{e}_{\pm 4}\,,\qquad S(\tilde{e}_{\pm 3}) = q^{\pm 3/2} e_{\pm 3}\,, \qquad S(\tilde{e}_{\pm 4}) = -q^{\pm 2} e_{\pm 4}\,. \tag{2.7}$$

The 14 generators $h_i, e_{\pm i}$ $(i = 1, 2)$ and $e_{\pm A}, \tilde{e}_{\pm A}$ $(A = 3, 4)$ form the antipode-extended Cartan–Weyl basis.

One can introduce the following class of standard involutions, satisfying (1.1):

(i) $|q| = 1$ $(\Delta^\pm \to \Delta^\pm; \lambda^2 = \epsilon^2 = 1)$:

$$k_i^+ = k_i \longrightarrow h_i^+ = -h_i\,,\qquad e_{\pm 1}^+ = \lambda e_{\pm 1}\,,\qquad e_{\pm 2}^+ = \epsilon e_{\pm 2}\,,\qquad e_{\pm 3}^+ = -\lambda\epsilon q^{\mp 1} e_{\pm 3}\,,\qquad e_{\pm 4}^+ = \epsilon q^{\mp 1} e_{\pm 4}\,. \tag{2.8a}$$

It has been shown in ref. [2] that the real quantum Poincaré algebras obtained by contraction of real forms of $U_q(\mathrm{Sp}(4;\mathbb{C}))$ will contain an undeformed O(2, 1) subalgebra. Such a real form is suitable for the description of the quantum deformation of the $D = 3$ conformal algebra (see ref. [14]).

(ii) q real ($\Delta^{\pm} \to \Delta^{\mp}; \lambda^2 = \epsilon^2 = 1$):

$$\begin{aligned} h_i^+ &= h_i & \longrightarrow \quad k_i^+ &= k_i\,, \\ e_{\pm 1}^+ &= \lambda e_{\mp 1} & e_{\pm 2}^+ &= \epsilon e_{\mp 2}\,, \\ e_{\pm 3}^+ &= -\epsilon\lambda q^{\pm 1}\widetilde{e}_{\mp 3} & e_{\pm 4}^+ &= \epsilon q^{\pm 1}\widetilde{e}_{\mp 4}\,. \end{aligned} \tag{2.8b}$$

The involutions (2.8) can be related with the class of $\oplus$-involutions considered in refs. [1,2] (and in particular with the one considered in ref. [1]) if we introduce the complex linear morphism Q of the $U_q\mathrm{Sp}(4;\mathbb{C})$ Hopf algebra, replacing q by q^{-1}, i.e.,[#5]

$$Q: \quad \begin{aligned} Qe_i &= e_i\,, & Qh_i &= h_i\,, \\ Qe_{\pm 3} &= -q^{\mp 1}\widetilde{e}_{\pm 3}\,, & Qe_{\pm 4} &= q^{\mp 1}\widetilde{e}_{\pm 4}\,, \\ Qq &= q^{-1}\,. \end{aligned} \tag{2.9}$$

Because Q is a $\oplus$-involution [see (1.3)] we have $+ \;=\; Q \circ \oplus$.

It can be shown that the involutions (2.8b) for $\lambda = \epsilon = 1$ describe the standard real Hopf algebra $U_q(\mathrm{O}(5))$, for $\epsilon = -\lambda = 1$ one obtains the standard real Hopf algebra $U_q(\mathrm{O}(4,1))$, and the remaining two choices ($\lambda = \epsilon = -1$ and $\lambda = -\epsilon = 1$) provide real standard Hopf algebras $U_q(\mathrm{O}(3,2))$. It appears that for our purpose only the choice $\lambda = -\epsilon = 1$ in (2.8b) is a proper one, providing after the contraction a quantum Poincaré algebra with undeformed O(3) rotations. This choice can be obtained by the multiplication by Q of the $\oplus$-involution used in ref. [1]. One gets the following standard +-involution:

$$\begin{aligned} e_1^+ &= e_{-1}\,, & e_2^+ &= -e_{-2}\,, \\ e_3^+ &= q\widetilde{e}_{-3}\,, & e_4^+ &= -q\widetilde{e}_{-4}\,. \end{aligned} \tag{2.10}$$

Because for $q = 1$ the morphism (2.9) describes an identity transformation, it follows from ref. [1] that the real form defined by the involution (2.10) describes the deformation of the O(3,2) Lie algebra, with the metric $g_{AB} = \mathrm{diag}(-1,1,1,1,-1)$.

It should be mentioned that the q-deformed generators M_{AB}, satisfying the condition $M_{AB} = M_{AB}^+$, can be introduced in many ways not necessarily leading to the Jacobi identities for the contracted quantum Poincaré algebra. We have performed the calculations for the following choice of the Cartan–Weyl basis for $U_q(\mathrm{O}(3,2))$ (q real):

$$M_3 = M_{12} = h_1\,, \qquad M_{\pm} = M_{23} \pm \mathrm{i}M_{31} = \sqrt{2}\,e_{\pm 1}\,,$$
$$L_3 = M_{34} = -(1/\sqrt{2})(q^{-\mathrm{i}/2}e_3 + q^{1+\mathrm{i}/2}\widetilde{e}_{-3})\,,$$

[#5] We found the morphism (2.9) denoted by σ in ref. [15], p. 37.

$$
\begin{aligned}
L_+ &= M_{14} + \mathrm{i}M_{24} = e_4 + e_{-2}, \\
L_- &= M_{14} - \mathrm{i}M_{24} = -(e_2 + q\tilde{e}_{-4}), \\
RP_0 &= M_{04} = h_3, \qquad RP_3 = M_{03} = (\mathrm{i}/\sqrt{2})(q^{1+\mathrm{i}/2}\tilde{e}_{-3} - q^{-\mathrm{i}/2}e_3), \\
RP_+ &= R(P_2 + \mathrm{i}P_1) = M_{02} + \mathrm{i}M_{01} = e_2 - q\tilde{e}_{-4}, \\
RP_- &= R(P_2 - \mathrm{i}P_1) = M_{02} - \mathrm{i}M_{01} = e_4 - e_{-2}.
\end{aligned} \tag{2.11}
$$

Because in the definitions (2.11) the generators $\tilde{e}_{-3}$, $\tilde{e}_{-4}$ enter in order to calculate the quantum algebra of the generators (2.11) one should supplement the algebra satisfied by the generators $h_i, e_{\pm i}$ $(i = 1,2)$, $e_{\pm 3}$ and $e_{\pm 4}$ (q-deformed Cartan–Weyl basis of $U_q(\mathrm{Sp}(4;\mathbb{C}))$ [1,2]) by the additional relations for the generators (2.6b). We have

$$
\begin{aligned}
[e_3, \tilde{e}_{-3}] &= (q^{-1} - q)q^{h_1}e_{-2}e_2 + (q - q^{-1})q^{-h_2}e_{-1}e_1 \\
&\quad + q^{h_1-h_2} - [h_1 + h_2 + 1], \\
[e_3, \tilde{e}_{-4}] &= (q - q^{-1})(q - 1)q^{-h_2}e_{-1}^2 e_1 \\
&\quad + (1 - q^2)q^{h_1}e_2\tilde{e}_{-3} + q^{-1}q^{-h_3}e_{-1}, \\
[e_4, \tilde{e}_{-3}] &= (q - q^{-1})(q^{-1} - 1)q^{-h_2}e_{-1}e_1^2 \\
&\quad + (q^{-2} - 1)q^{h_1}e_3e_{-2} - q^{-1}q^{-h_3}e_1, \\
[e_4, \tilde{e}_{-4}] &= (q - q^{-1})(q^{-1} - 1)q^{-h_2}\{qe_{-1}^2e_1^2 - e_{-1}e_1^2e_{-1}\} \\
&\quad + (q^{-1} - q)q^{h_1}e_3\tilde{e}_{-3} + (1 - q^2)q^{2h_1}e_2e_{-2} \\
&\quad + (q - q^{-1})q^{2h_1}e_{-2}e_2 + (q^{-1} - q)q^{h_1-h_2}e_{-1}e_1 \\
&\quad + q^{-1}q^{-h_3}[h_1] - q^{2h_1-h_2} + q^{h_1}[h_1 + h_2 + 1].
\end{aligned} \tag{2.12}
$$

The commutation relations for two tilde generators can be obtained from the formulae in ref. [1] by the action of the antipode, e.g.,

$$
[\tilde{e}_3, \tilde{e}_4] = -q^{7/2}S([e_3, e_4]) = q^{-7/2}(1 - q)S(e_4e_3) = (q - 1)\tilde{e}_3\tilde{e}_4. \tag{2.13}
$$

Calculation of coproducts for physical generators requires the knowledge of the following formulae:

$$
\begin{aligned}
\Delta(\tilde{e}_3) &= \tilde{e}_3 \otimes q^{h_3/2} + q^{h_3/2} \otimes \tilde{e}_3 + (q^{-1} - q)q^{-h_1/2}e_2 \otimes e_1q^{h_2/2}, \\
\Delta(\tilde{e}_{-3}) &= \tilde{e}_{-3} \otimes q^{h_3/2} + q^{-h_3/2} \otimes \tilde{e}_{-3} - (q^{-1} - q)q^{-h_2/2}e_{-1} \otimes e_{-2}q^{h_1/2}, \\
\Delta(\tilde{e}_4) &= \tilde{e}_4 \otimes q^{h_4/2} + q^{-h_4/2} \otimes \tilde{e}_4 + (q - q^{-1}) \\
&\quad \times \{(1 - q^{-1})q^{-h_1}e_2 \otimes e_1^2q^{h_2/2} - q^{-h_1/2}\tilde{e}_3 \otimes e_1q^{h_3/2}\}, \\
\Delta(\tilde{e}_{-4}) &= \tilde{e}_{-4} \otimes q^{h_4/2} + q^{-h_4/4} \otimes \tilde{e}_{-4} + (q - q^{-1}) \\
&\quad \times \{(q - 1)q^{-h_2/2}e_{-1}^2 \otimes e_{-2}q^{h_1} + q^{-h_3/2}e_{-1} \otimes \tilde{e}_{-3}q^{h_1/2}\}.
\end{aligned} \tag{2.14}
$$

3. The contraction to the standard real quantum Poincaré algebra

In order to obtain the standard real κ-Poincaré algebra we proceed further as follows:

(i) Using the formulae for the commutators and coproducts of the antipode-extended Cartan–Weyl basis and the deformations (3.11) we can write the q-deformation of the O(3, 2) Lie algebra as well as the coproduct relations for the q-deformed O(3, 2) generators.

(ii) We perform further the quantum de Sitter contraction, obtained by the conventional rescaling of the O(3, 2) rotation generators,

$$\begin{aligned} &M_{\mu\nu} \text{ unchanged} \qquad && (M_{\mu\nu}^{+} = M_{\mu\nu}), \\ &M_{4\mu} = RP_{\mu} && (P_{\mu}^{+} = P_{\mu}), \end{aligned} \tag{3.1}$$

and the $q \to 1$ limit described by (1.2).

As a result we obtain the following q-deformed Poincaré algebra:

(a) *Three-dimensional* O(3) *rotations* ($M_{\pm} = M_1 + \mathrm{i}M_2 \equiv M_{23} \pm \mathrm{i}M_{31}$; $M_3 = M_{12}$);

(i) commutation relations:

$$[M_+, M_-] = 2M_3, \qquad [M_3, M_{\pm}] = \pm M_{\pm}; \tag{3.2a}$$

(ii) coproducts:

$$\Delta M_i = M_i \otimes I + I \otimes M_i; \tag{3.2b}$$

(iii) antipode:

$$S(M_i) = -M_i. \tag{3.2c}$$

(b) *Boosts sector* O(3, 1) ($L_{\pm} = M_{14} \pm \mathrm{i}M_{24}$, $L_3 = M_{34}$);

(i) commutation relations:

$$\begin{aligned} [L_+, L_-] &= -2M_3 \cosh(P_0/\kappa) + \frac{1}{2\kappa^2} P_3^2 \\ &\quad + \frac{1}{\kappa^2} M_3 P_3^2 - \sinh(P_0/\kappa), \\ [L_+, L_3] &= \mathrm{e}^{-P_0/\kappa} M_+ + \frac{1}{2\kappa} (\mathrm{i}P_3 L_+ + L_3 P_-) \\ &\quad - \frac{\mathrm{i}}{2\kappa^2} M_3 P_3 P_- + \frac{1}{4\kappa^2} (2 - \mathrm{i}) P_3 P_-, \\ [L_-, L_3] &= -\mathrm{e}^{-P_0/\kappa} M_- + \frac{1}{2\kappa} (\mathrm{i}L_- P_3 - P_+ L_3) \\ &\quad - \frac{\mathrm{i}}{2\kappa^2} P_3 P_+ M_3 - \frac{1}{4\kappa^2} (2 + \mathrm{i}) P_3 P_+, \end{aligned} \tag{3.3a}$$

$$[M_3, L_3] = 0, \qquad [M_3, L_\pm] = \pm L_\pm,$$
$$[M_\pm, L_\pm] = \mp\frac{1}{2\kappa}M_\pm P_\mp,$$
$$[M_+, L_-] = 2L_3 - \frac{1}{2\kappa}P_+M_+ + \frac{i}{\kappa}M_3P_3 + \frac{1}{\kappa}P_3,$$
$$[M_-, L_+] = -2L_3 + \frac{1}{2\kappa}M_-P_- + \frac{i}{\kappa}P_3M_3 - \frac{1}{\kappa}P_3,$$
$$[M_+, L_3] = -L_+ + \frac{1}{2\kappa}M_3P_- + \frac{i}{2\kappa}P_-,$$
$$[M_-, L_+] = L_- - \frac{1}{2\kappa}P_+M_3 + \frac{i}{2\kappa}P_+; \tag{3.3b}$$

(ii) coproducts:

$$\Delta L_3 = L_3 \otimes e^{P_0/2\kappa} + e^{-P_0/2\kappa} \otimes L_3 + \frac{1}{2\kappa}e^{-P_0/2\kappa}(M_+ \otimes P_+ + M_- \otimes P_-),$$
$$\Delta L_\pm = L_\pm \otimes e^{P_0/2\kappa} + e^{-P_0/2\kappa} \otimes L_\pm + \frac{1}{2\kappa}\left(P_\mp \otimes M_3 e^{P_0/2\kappa} - e^{-P_0/2\kappa}M_3 \otimes P_\mp\right)$$
$$\mp \frac{i}{\kappa}e^{-P_0/2\kappa}M_\pm \otimes P_3; \tag{3.3c}$$

(iii) antipode:

$$S(L_3) = -L_3 + \frac{i}{2\kappa}P_3 + \frac{1}{2\kappa}(M_+P_+ + M_-P_-),$$
$$S(L_\pm) = -L_\pm \mp \frac{1}{\kappa}P_\mp \mp \frac{i}{\kappa}M_\pm P_3. \tag{3.3d}$$

(c) *Translations sector* ($P_\pm = P_2 \pm iP_1, P_3, P_0$);

(i) commutation relations:

$$[P_\mu, P_\nu] = 0 \quad (\mu, \nu = 0, 1, 2, 3),$$
$$[M_i, P_j] = i\epsilon_{ijk}P_k, \qquad [M_i, P_0] = 0, \tag{3.4a}$$

$$[L_3, P_0] = iP_3, \qquad [L_3, P_3] = i\kappa\sinh(P_0/\kappa) - \frac{i}{2\kappa}P_+P_-,$$
$$[L_3, P_2] = \frac{i}{2\kappa}P_3P_2, \qquad [L_3, P_1] = \frac{i}{2\kappa}P_3P_1,$$
$$[L_\pm, P_0] = iP_1 \mp P_2, \qquad [L_\pm, P_2] = \mp\kappa\sinh(P_0/\kappa) \pm \frac{1}{2\kappa}P_3^2,$$
$$[L_\pm, P_3] = \mp\frac{1}{2\kappa}P_3P_\mp, \qquad [L_\pm, P_1] = i\kappa\sinh(P_0/\kappa) + \frac{1}{2i\kappa}P_3^2; \tag{3.4b}$$

(ii) coproducts:

$$\Delta P_0 = P_0 \otimes I + I \otimes P_0,$$
$$\Delta P_i = P_i \otimes e^{P_0/2\kappa} + e^{-P_0/2\kappa} \otimes P_i \quad (i = 1, 2, 3); \tag{3.4c}$$

(iii) antipode:

$$S(P_\mu) = -P_\mu . \tag{3.4d}$$

Following ref. [10] we introduce a nonlinear transformation of the boost generators,

$$\begin{aligned}
\widetilde{L}_+ &= L_+ + \frac{\mathrm{i}}{2\kappa} M_+ P_3 - \frac{1}{2\kappa} P_- , \\
\widetilde{L}_- &= L_- - \frac{\mathrm{i}}{2\kappa} P_3 M_- - \frac{1}{2\kappa} P_+ , \\
\widetilde{L}_3 &= L_3 - \frac{\mathrm{i}}{4\kappa} (M_+ P_+ + P_- M_-) + \frac{1}{2\kappa} P_3 ,
\end{aligned} \tag{3.5}$$

simplifying the κ-Poincaré algebra substantially. The new boosts satisfy the following relations:

$$\begin{aligned}
&[M_l, \widetilde{L}_j] = \mathrm{i}\epsilon_{ljk} \widetilde{L}_k , \\
&[P_0, \widetilde{L}_k] = -\mathrm{i}P_k , \qquad [P_k, \widetilde{L}_j] = -\mathrm{i}\kappa\delta_{kj} \sinh(P_0/\kappa) , \\
&[\widetilde{L}_l, \widetilde{L}_j] = -\mathrm{i}\epsilon_{lkj} \left(M_k \cosh(P_0/\kappa) - \frac{1}{4\kappa^2} P_k \, (\boldsymbol{P} \cdot \boldsymbol{M}) \right) .
\end{aligned} \tag{3.6}$$

It is interesting to observe that the algebra (3.6) differs from the one obtained in ref. [10] only by the replacement $\kappa \to \mathrm{i}\kappa$. The same holds for the coproduct formulae,

$$\begin{aligned}
\Delta(\widetilde{L}_i) &= \widetilde{L}_i \otimes \mathrm{e}^{P_0/2\kappa} + \mathrm{e}^{-P_0/2\kappa} \otimes \widetilde{L}_i \\
&\quad + \frac{1}{2\kappa} \epsilon_{ijk} \left(P_j \otimes M_k \mathrm{e}^{P_0/2\kappa} - \mathrm{e}^{-P_0/2\kappa} M_j \otimes P_k \right) .
\end{aligned} \tag{3.7}$$

The coproducts (3.7) which satisfy the relation (1.1) permit us to define the tensor product representations in Hilbert space. For completeness we give also the antipodes:

$$S(\widetilde{L}_i) = -\widetilde{L}_i + \frac{3\mathrm{i}}{2\kappa} P_i . \tag{3.8}$$

4. The Casimirs and the κ-deformed Heisenberg–Poincaré algebra

One can construct the quantum deformation of the quadratic Casimir, describing the quantum relativistic mass square operator. One gets

$$C_1 = P_1^2 + P_2^2 + P_3^2 + 2\kappa^2 \left[1 - \cosh(P_0/\kappa) \right] = \boldsymbol{P}^2 - \left[2\kappa \sinh(P_0/2\kappa) \right]^2 . \tag{4.1}$$

It should be mentioned that recently the $D = 4$ mass square Casimir was proposed in refs. [9,16] as the extension of the results obtained for $D = 3$ Poincaré algebra. We can say therefore that the deformed square mass formula of refs. [9,16] finds full theoretical justification in this paper.

The second Casimir can be obtained by introducing the κ-deformed Pauli–Lubanski four-vector

$$W_0 = \boldsymbol{P} \cdot \boldsymbol{M} ,$$
$$W_k = \kappa M_k \sinh(P_0/\kappa) + \epsilon_{kij} P_i \widetilde{L}_j , \tag{4.2}$$

where $\widetilde{L}_i$ is defined by formulae (3.5).

Writing the commutators of W_μ with themselves and $\widetilde{L}_i$ one obtains the relations, presented in ref. [10], modified only by the replacement $\kappa \to \mathrm{i}\kappa$. The formula for the second Casimir takes the form

$$C_2 = \left[\cosh(P_0/\kappa) - \boldsymbol{P}^2/4\kappa^2\right] W_0^2 - \boldsymbol{W}^2 . \tag{4.3}$$

Let us write down the spinless realization of the κ-Poincaré algebra, for which $\boldsymbol{P} \cdot \boldsymbol{M} = 0$ [#6]

$$P_\mu = \frac{1}{\mathrm{i}} \frac{\partial}{\partial x_\mu} , \tag{4.4a}$$

$$M_i = \epsilon_{ijk} x_j P_k , \qquad \widetilde{L}_i = x_0 P_i - x_i \widetilde{P}_0 , \tag{4.4b}$$

where

$$\widetilde{P}_0(P_0) = \kappa \sinh(P_0/\kappa) = \frac{1}{2a}\left(\mathrm{e}^{aP_0} - \mathrm{e}^{-aP_0}\right) , \quad a = \frac{1}{\kappa} , \tag{4.5}$$

we see that we obtain a new type of realizations containing derivatives of arbitrarily high orders. The κ-deformed boosts $\boldsymbol{L}_i$ act explicitly on the scalar field $\phi(\boldsymbol{x}, t)$ as follows: [#7]

$$\widehat{L}_i \phi(\boldsymbol{x}, t) = x_0 (\partial/\partial x_i) \phi(\boldsymbol{x}, t) - \tfrac{1}{2}\kappa x_i (\phi(\boldsymbol{x}, t + \mathrm{i}/\kappa) - \phi(\boldsymbol{x}, t - \mathrm{i}/\kappa)) . \tag{4.6}$$

We see that a new element occurring in our theory is the symmetric finite difference derivative in the imaginary time direction, represented by the operator $\widetilde{P}_0$. It should be stressed that this finite difference derivative is different from the so-called q-derivative [18] used in the differential realization of Drinfeld–Jimbo q-deformations of simple Lie algebras (see, e.g., ref. [19]). Using the formulae (4.5) one can calculate the coproduct

$$\Delta(\widetilde{P}_0) = \widetilde{P}_0(\Delta(P_0)) = \frac{1}{2a}(\mathrm{e}^{aP_0} \otimes \mathrm{e}^{aP_0} - \mathrm{e}^{-aP_0} \otimes \mathrm{e}^{-aP_0})$$
$$= \widetilde{P}_0 \otimes \mathrm{e}^{aP_0} + \mathrm{e}^{-aP_0} \otimes \widetilde{P}_0 . \tag{4.7}$$

Formula (4.7) describing the Leibniz rule for the finite difference derivative (4.5) shows clearly that the change of the generator $P_0 \to \widetilde{P}_0$ does not eliminate the presence of P_0 in the coproduct formulae.

[#6] The realization (3.8) was firstly obtained independently by P. Zaugg and the Lodz group [10]. We were informed [17] that the realization (4.8) was extended by the Lodz group to arbitrary spin.

[#7] For the quantum Poincaré algebra given in ref. [1] the shift of the time arguments is real.

The relations (4.4a, b) can be used for the extension of the κ-Poincaré algebra by covariant four-coordinates $\hat{X}_\mu$, satisfying the relation

$$[P_\mu, \hat{X}_\nu] = -\mathrm{i}g_{\mu\nu} . \tag{4.8}$$

We obtain

$$\begin{aligned} [M_i, \hat{X}_j] &= \mathrm{i}\epsilon_{ijk}\hat{X}_k , \qquad && [M_i, \hat{X}_0] = 0 , \\ [\widetilde{L}_i, \hat{X}_j] &= -\mathrm{i}\delta_{ij}\hat{X}_0 , \qquad && [\widetilde{L}_i, \hat{X}_0] = \mathrm{i}\cosh(P_0/\kappa)\, \hat{X}_i . \end{aligned} \tag{4.9}$$

The formulae (4.8), (4.9) extend the κ-deformed Poincaré algebra to the κ-deformed Heisenberg–Poincaré algebra. It would be interesting to find the extension of coproduct formulae [see (3.2b), (3.4c) and (3.7)] promoting the κ-deformed Heisenberg–Poincaré algebra to a real Hopf algebra with fourteen generators $P_\mu, \hat{X}_\mu, M_{\mu\nu}$.

5. Final remarks

The aim of this lecture is to present a new quantum real Poincaré algebra, with standard reality condition having properties given by (1.1). In such a way the problems related to the choice of $\oplus$-involution in refs. [1–4] are avoided. In particular the composition law of two three-momenta leads to the following real value of the total four-momentum [see also (3.4c)]:

$$\begin{aligned} P_i^{(1+2)} &= P_i^{(1)} \exp(P_0^{(2)}/2\kappa) + P_i^{(2)} \exp(-P_0^{(1)}/2\kappa) \\ &= P_i^{(1)} + P_i^{(2)} + \frac{1}{2\kappa}(P_i^{(1)}P_0^{(2)} - P_i^{(2)}P_0^{(1)}) + \mathrm{O}(1/\kappa^2) , \end{aligned} \tag{5.1}$$

with the classical addition formula valid for $P_0^{(i)} \ll \kappa$. The existence of the coproduct implies the existence of the tensor product of representations, i.e. one can pass from irreducible representations in Hilbert space (quantum mechanics) to the reducible representations in Fock space (free quantum fields).

The consequences of the composition law (5.1) for the three-momenta in classical mechanics and field theory are now under consideration.

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